Inverse estimation of the elastic and anelastic properties of the porous frame of anisotropic open-cell foams

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This paper presents a method for simultaneously identifying both the elastic and anelastic properties of the porous frame of anisotropic open-cell foams. The approach is based on an inverse estimation procedure of the complex stiffness matrix of the frame by performing a model fit of a set of transfer functions of a sample of material subjected to compression excitation *in vacuo*. The material elastic properties are assumed to have orthotropic symmetry and the anelastic properties are described using a fractional-derivative model within the framework of an augmented Hooke's law. The inverse estimation problem is formulated as a numerical optimization procedure and solved using the globally convergent method of moving asymptotes. To show the feasibility of the approach a numerically generated target material is used here as a benchmark. It is shown that the method provides the full frequency-dependent orthotropic complex stiffness matrix within a reasonable degree of accuracy. © *2012 Acoustical Society of America*.
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I. INTRODUCTION

Porous wools and foams exhibit varying degrees of anisotropy in their dynamic macroscopic elastic and anelastic properties. Such anisotropy may be attributed to the microstructural geometric shape of the cells,[1,2] which in the case of foams arises from the directions of injection flow and rise of the material during the manufacturing process. Furthermore, the anelastic damping properties are also influenced by the chemical formulation of the polymer used for the solid material constituting the frame of the porous material, as well as by the state of deformation in the frame.[3]

In many applications involving porous materials, the assumption of isotropic properties yields satisfactory correlations between experimental and computed results. This is particularly true in cases where airborne sound absorption is of interest. However, for porous models in the scope of classical vibroacoustics, and specifically those where the structure-borne properties are important, the sources of differences between predicted and measured results are not fully understood. A not yet fully explored aspect is whether the possible anisotropy of the constitutive properties may be important enough to influence the behavior to a significant extent. Therefore, detailed studies are necessary in order to assess the influence of anisotropy on the vibroacoustic behavior of structures comprising porous materials.

The problem of identifying the elastic moduli of porous materials has been studied previously by one of the authors in collaborative efforts initially focused on fibrous materials.[4,5] In those works, the viscoelasticity was assumed to be represented by an isotropic material model, thus not necessarily following the material symmetries of the elasticity model. Alternative approaches for the determination of the elastic parameters of open-cell foams consist of quasi-static testing to identify the elastic modulus and the Poisson's ratio.[6–9] In previous work on the mechanical characterization of anisotropic foams, Melon *et al.*[6] assumed that the mounting of the cubic foam sample tested was aligned with the rise direction. They found that the tested foams were reasonably well described by a transversely isotropic model, in particular in terms of the Young's moduli. However, the estimation of the shear moduli was reported to be less accurate. An exhaustive review of the existing methods for characterizing porous materials would be out of the scope of the present paper and the reader is referred to the recently published work by Jaouen *et al.*[10] Here it will suffice to recall one of the main conclusions drawn therein stressing the importance of the work discussed in the present paper, i.e., the lack of techniques for accessing the elastic and damping properties of anisotropic porous media.

A complicating aspect in the characterization of porous materials is that the interstitial fluid in the pores and its interaction with the solid frame contribute to the dissipation of energy. Thus, to establish a constitutive model, the dissipative mechanisms related to internal frequency-dependent fluid-structure interaction must be isolated from the dissipation related to the anelastic relaxation mechanisms associated with the deformation state of the solid frame. Early works by Pritz[11–14] were focused on the dynamic characterization of fibrous materials and foams using vibration testing in a seismic mass setup. The measurements were performed in a vacuum chamber in order to remove the effect of air in the pores of the material.

The key point of interest for the current work is the consideration of the dynamic elastic moduli of the solid frame as the superposition of elastic and anelastic contributions. In fact, the elasticity is frequency independent and the anelasticity is responsible for the variation of the dynamic properties with frequency. Several equivalent forms of the parameterization of

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such anelastic part may be used. Biot introduced the concept of hidden thermodynamic variables,[15] later extended by Dovstam and Dalenbring to a discrete multiple-process augmented Hooke's law.[16–19] Alternative parameterizations have been discussed by several authors,[20–22] providing a compact expression with fewer parameters. In particular, the fractional derivative model,[23,24] provides a causal description of the damping mechanisms. It consists in a generalization of the classical unit-order time derivative, from the consideration of energy dissipation as a memory effect in the deformation of the material. For instance, Pritz[25,26] showed that a fractional-derivative model described by four or five parameters allows to predict the response of isotropic viscoelastic materials and to extract their properties from experimental data. Recent work[27,28] shows the applicability of a fractional-derivative model to the inverse estimation of elastic and anelastic properties of isotropic materials and of viscoelastic laminated plates.

It should be noted that in the context of dynamical characterization of a certain material, the unique constitutive properties are the dynamic moduli at the frequencies of interest. The actual variation with frequency may then be parameterized in any suitable way depending on the specific purpose at hand,[22] bearing in mind that such parameterization is not unique. Furthermore, the elastic anisotropy and the anelastic anisotropy do not necessarily share the same type of symmetry, a circumstance that considerably complicates any attempt to characterize a real unknown material.

The current paper proposes the first steps in a methodology for the identification of the anisotropic elastic and anelastic moduli of the porous frame of open-cell foams, through an inverse estimation approach. The constitutive properties of the frame are modeled using an augmented Hooke's law, by considering the stiffness matrix as a superposition of orthotropic elastic and anelastic contributions. The inverse estimation procedure consists of a fit of numerically simulated frequency responses onto the responses of the targeted material, with the constitutive properties used as the variables of the adaptation. This requires an experimental setup which is simple enough to be accurately represented in a numerical simulation model. Here, the basic configuration used by Pritz is chosen, in which a sample of material is located in a vacuum chamber, placed on a shaker and loaded with a seismic mass. The vertical vibration transfer functions between the shaker and the surface of the seismic mass are then used as targets for the inverse estimation. Special care is taken to extract both compressive and shear moduli, which is done through a variation of the loading conditions used. The present paper is intended to discuss the theoretical aspects of the proposed method. Therefore, the experimental data used as a target is fictitious and free from imperfections and noise. In a forthcoming paper, the application of the method to a real foam will be discussed.

II. MATERIAL MODEL

A. Augmented Hooke's law

In order to characterize the *in vacuo* dynamical properties of a porous material, it is necessary to establish a model for describing the elasticity of the porous frame without the influence of the fluid filling the pores. The starting point is taken in the constitutive relations for anisotropic fluid-filled porous materials given by Biot's equations.[29–31] Considering zero pressure in the fluid phase in those equations leads to a Hooke's law for the frame, which can be written in the frequency domain as

$$\sigma_i(\omega) = H_{ij}(\omega)\varepsilon_j(\omega), \quad i,j = 1,\dots,6, \tag{1}$$

where σ_i and ε_j contain the components of the stress and strain tensors according to

$$\boldsymbol{\sigma} = [\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23}\ \sigma_{31}\ \sigma_{12}]^{\mathrm{T}}, \tag{2}$$

$$\varepsilon = [\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ 2\varepsilon_{23}\ 2\varepsilon_{31}\ 2\varepsilon_{12}]^{\mathrm{T}}, \tag{3}$$

and H_{ij} denote the components of the stiffness matrix, which entirely characterizes the material.

As discussed in the Introduction, several different models for such a dynamic stiffness matrix are available in literature, in particular when including viscoelastic effects. For the purposes of the present paper, the augmented Hooke's law[22] is used. This has the advantage of providing a causal, functional description of the deformation of the material and allows separation of the stiffness matrix into an elastic part, related to fully relaxed material deformation, and an anelastic part, accounting for reversible viscoelastic deformation, i.e., anelastic deformation. With this choice of separation of the moduli, into terms which are respectively independent and dependent on the frequency, several equivalent forms of the parameterization of the viscoelastic part of the stiffness matrix may be used as discussed above. Previous work, i.e., Refs. 4 and 5 used the discrete multiple process augmented Hooke's law[16] and for the present study it was decided to explore the advantages with the fractional derivative model.

The components of the stiffness matrix assume the form[22]

$$H_{ij}(\omega) = H_{ij}^{(0)} + \tilde{H}_{ij}(\omega), \tag{4}$$

where $H_{ij}^{(0)}$ and $\tilde{H}_{ij}(\omega)$ describe the elastic and anelastic effects in the material, respectively. Using a sufficiently general form of the fractional-derivative-based frequency dependence, each component of the viscoelastic part of the stiffness matrix then assumes the form

$$\tilde{H}_{ij}(\omega) = \frac{b_{ij}\cdot s^{\alpha_{ij}}}{\beta_{ij} + s^{\alpha_{ij}}}, \tag{5}$$

where $s = i\omega$ denotes Laplace's variable, α_{ij} is the fractional derivative order for the H_{ij} modulus, $\beta_{ij} = 1/\tau_{ij}$ is the relaxation frequency, τ_{ij} being the relaxation time for the H_{ij} modulus of the material, and b_{ij} is a real matrix giving the magnitude of the contribution of viscoelastic effects to the motion of the material.

The derivative order is within the range

$$0 < \alpha_{ij} \leq 1, \quad i,j = 1,\dots,6 \tag{6}$$

the limiting case $\alpha_{ij} = 1$ corresponding to the conventional model of dissipation effects.

It is quite obvious that the general model given above poses a formidable challenge in finding its parameters. Recalling that the primary objective of the current paper is to verify the feasibility of a method for identifying the properties of the porous frame, some reasonable restrictions to the model of the frequency dependence are introduced as a first attempt. First, it is assumed that the fractional derivative order is the same for all moduli, i.e., $\alpha_{ij} = \alpha$, and that the same holds for the relaxation frequencies, $\beta_{ij} = \beta$. Then it is assumed that the elastic and the anelastic moduli are collinear. This can be expressed by stating

$$b_{ij} = b \cdot H_{ij}^{(0)} \tag{7}$$

and yields the simplified augmented Hooke's law

$$H_{ij}(\omega) = H_{ij}^{(0)}\left(1 + \frac{b \cdot s^{\alpha}}{\beta + s^{\alpha}}\right), \tag{8}$$

which is an approximation that actually restricts the modeling to represent the classical case of proportional damping. As the approach discussed is quite general, it should be pointed out that this approximation does not restrict the domain of applicability of the method itself. Furthermore, the parameterization of the stiffness matrix provided by Eq. (8) is equivalent to the four-parameter model proposed by Pritz,[25] whose only restriction is that it is applicable to materials with a symmetric loss peak.[26] Choosing different forms of either parameterization or material symmetries will require a larger computational effort but would not necessarily bring any further useful information for the proof of concept discussed in the present paper.

An important aspect of this type of parameterization is that it provides a compact representation of the frequency dependency of the material moduli. As stated in Sec. I, the separation between static, elastic and frequency dependent, anelastic constitutive properties in Eq. (4) is to a certain extent arbitrary. The unique properties are the actual dynamic moduli in $H_{ij}(\omega)$ and this fact gives some freedom in the inverse estimation process setup which will be described below.

B. Quantifying the degree of anisotropy of the material

In order to facilitate the inverse estimation, it was found useful to express the stiffness matrix in terms of a number of scaling constants describing the degree of anisotropy of the material with respect to an arbitrarily chosen isotropic, baseline material. As discussed above this can be done without any loss of generality and these scaling constants are then used as the set of parameters on which the estimation procedure is based, as described later-on in Sec. III. Here, the reformulation in terms of these scaling constants starts from the assumption that the material under consideration is assumed to exhibit orthotropic symmetry, in which case the compliance matrix, i.e., the inverse of the stiffness matrix, can be written as[32]

$$S_{ij} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}, \tag{9}$$

where E_k is the Young's modulus along axis k, G_{kl} is the shear modulus in plane (k, l) and ν_{kl} is the Poisson's ratio for stress along k resulting in transverse strain along l. Furthermore, the symmetry of the compliance matrix yields

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}. \tag{10}$$

It is well known that the stiffness and compliance matrices must be positive-definite, a fact that may be used as constraints on the physical parameters in the estimation, as[32]

$$\begin{cases} E_k > 0, & \text{(11a)} \\ G_{kl} > 0, & \text{(11b)} \\ \nu_{ij}^2 < \frac{E_i}{E_j}, & \text{(11c)} \\ 2\nu_{21}\nu_{32}\nu_{13} < 1 - \frac{E_1}{E_2}\nu_{21}^2 - \frac{E_2}{E_3}\nu_{32}^2 - \frac{E_3}{E_1}\nu_{13}^2. & \text{(11d)} \end{cases}$$

As a means of quantifying the degree of anisotropy of the material, the parameters E_k, G_{kl} and ν_{kl} are written in terms of the properties of the isotropic, baseline material and a number of scaling constants, as

$$\begin{cases} E_m = \xi_m E, \; m = 1, 2, 3, & \text{(12a)} \\ G_{kl} = \xi_m G, \; m = 9 - k - l = 4, 5, 6, & \text{(12b)} \\ \nu_{kl} = \xi_m \nu, \; m = 4 + k + l = 7, 8, 9, & \text{(12c)} \end{cases}$$

where ξ_m are scaling constants for each of the nine parameters and E, $G = E/2(1+\nu)$ and ν are, respectively, the Young's modulus, the shear modulus and the Poisson's ratio of the baseline isotropic medium, with $E > 0$ and $-1 < \nu < 1/2$.

Using these relations, Eq. (9) can then be written in terms of the scaling constants ξ_m, $m = 1, \ldots, 9$ as

$$S_{ij} = \begin{bmatrix} \frac{1}{\xi_1 E} & -\frac{\xi_7 \nu}{\xi_1 E} & -\frac{\xi_8 \nu}{\xi_1 E} & 0 & 0 & 0 \\ & \frac{1}{\xi_2 E} & -\frac{\xi_9 \nu}{\xi_2 E} & 0 & 0 & 0 \\ & & \frac{1}{\xi_3 E} & 0 & 0 & 0 \\ & & & \frac{1}{\xi_4 G} & 0 & 0 \\ & \text{(sym)} & & & \frac{1}{\xi_5 G} & 0 \\ & & & & & \frac{1}{\xi_6 G} \end{bmatrix}. \tag{13}$$

The set of parameters ξ_m determine the degree of anisotropy of the material, such that the case $\xi_m = 1\,\forall m$ corresponds to the baseline isotropic material. The validity of Eq. (13) is restricted by the condition given in Eqs. (11), which in turn can be written as

$$\begin{cases} \xi_i > 0,\ i = 1, ..., 6, & \text{(14a)} \\ \xi_{4+i+j}^2 \nu^2 < \dfrac{\xi_i}{\xi_j},\ ij = 21, 13, 32, & \text{(14b)} \\ 2\nu^3 \xi_7 \xi_8 \xi_9 < 1 - \nu^2 \displaystyle\sum_{ij=21,13,32} \frac{\xi_j}{\xi_i} \xi_{4+i+j}^2. & \text{(14c)} \end{cases}$$

In addition to the scaling constants used for the nine elastic moduli of the material, three additional dimensionless quantities are used in order to relate the anelastic properties of the orthotropic unknown material to the ones of the isotropic material of reference, by writing

$$\begin{cases} \alpha = \xi_{10}\alpha_0, & \text{(15a)} \\ \beta = \xi_{11}\beta_0, & \text{(15b)} \\ b = \xi_{12} b_0, & \text{(15c)} \end{cases}$$

where α_0, β_0 and b_0 are the anelastic properties of the baseline material.

The variable change using scaling constants as a basis for the estimation of the properties of the material presents two main advantages. The first is that the set of unknown scaling constants is dimensionless, thus rendering the estimation procedure less sensitive to a higher degree of anisotropy in one or several moduli as well as automatically providing a proper scaling of the parameters used in the optimization procedure, thus avoiding possible numerical difficulties in the calculation of gradients and convergence. Another advantage is that the isotropic baseline material can be chosen as an equivalent medium describing the mean isotropic properties of the actual material, if its degree of anisotropy is low. Such an equivalent isotropic medium can either be arbitrarily chosen or obtained by existing inverse methods for isotropic foams, e.g., Refs. 6 and 33. In that sense, the inverse estimation method presented herein will provide the deviation from the equivalent isotropic material by means of the scaling constants and does as such result in a refinement of the properties estimated by existing techniques. For a given material, the pertinence of such a refined model depends on how deviated is the refinement with respect to the isotropic baseline material, which can be estimated, e.g., by using the formalism proposed by Norris.[34]

III. INVERSE ESTIMATION PROCEDURE

A. Optimization problem to solve

The basis for the proposed inverse estimation approach is a recorded set of target frequency response functions obtained for the as yet unknown material. Using an appropriate simulation model, the unknown constitutive parameters are varied such that a satisfactory fit of a predicted frequency response to the target frequency response functions is obtained. Thus, an optimal set of constitutive parameters is retained when the simulated frequency response data is close to the target frequency response data, within a prescribed tolerance. As outlined above, the variables used in the current work are the scaling factors defined in Eq. (12). In the following, the objective function and the constraint functions of the optimization problem are defined.

The aim of the optimization procedure is to find the twelve unknown elastic and anelastic parameters of the material. As a preliminary step in the estimation of the parameters of the model, the isotropic baseline material is defined by arbitrarily setting numerical values for the parameters E, ν, α_0, β_0 and b_0. The choice of such parameters must represent a feasible material, within the requirements expressed above, i.e., $E > 0$, $-1 < \nu < 1/2$ and $0 < \alpha_0 \le 1$. The optimization space is then defined as the set of unknown constitutive parameters, here the scaling constants previously defined, written as

$$\mathbf{x} = [\,\xi_1 \quad \xi_2 \quad \xi_3 \quad \xi_4 \quad \xi_5 \quad \xi_6 \quad \xi_7 \quad \xi_8 \quad \xi_9 \quad \xi_{10} \quad \xi_{11} \quad \xi_{12}\,]^{\mathrm{T}}, \tag{16}$$

where, for clarity it is recalled that, the different parameters used, respectively, denote the non-dimensional counterparts of

$$\mathbf{X} = [E_1 \quad E_2 \quad E_3 \quad G_{23} \quad G_{31} \quad G_{12} \quad \nu_{12} \quad \nu_{13} \quad \nu_{23} \quad \alpha \quad \beta \quad b]^{\mathrm{T}}. \tag{17}$$

The model fit is performed on a set of transfer functions, denoted u, to be specified later on in Sec. III B. The target variable space is denoted $\mathbf{x}_0$, the aim being to obtain $\mathbf{x} \approx \mathbf{x}_0$ by minimizing an appropriate objective function. The objective function to be minimized is defined as the quadratic relative difference between the simulated transfer functions and the target transfer functions, summed over the frequency points, as

$$f_0(\mathbf{x}) = 1 + \sum_{p=1}^{N} \left| \frac{u(\omega_p, \mathbf{x}) - u(\omega_p, \mathbf{x}_0)}{u(\omega_p, \mathbf{x}_0)} \right|^2, \tag{18}$$

where ω_p is the discrete circular frequency axis on which the computations are performed and N is the number of frequency points. Note that in Eq. (18) a constant is added to the objective function in order to avoid numerical problems for small values.

Furthermore, the optimization problem is subject to a number of numerical constraints, corresponding to the physical requirements of the material given by Eqs. (14).

B. Setup

The choice of a suitable setup is to some extent a matter of taste and convenience. The only restriction is that the state of deformation, the boundary conditions imposed and the excitation itself should be representable with a high degree of accuracy in a simulation model. Many possible configurations satisfy these conditions and allow the relevant vibration data to be recorded and thus serve as the target response for extracting the unknown properties of a given material.

The aim here is to design a realistic measurement setup that is sensitive to the different types of motion of the material, which are related to the moduli to estimate. The setup chosen here represents an experimental setup currently under development at the Marcus Wallenberg Laboratory for Sound and Vibration Research of the Royal Institute of Technology (KTH).[35] It utilizes a cubic sample of material placed between a plate driven by a vertical shaker at the bottom and a seismic mass on the top, as depicted in Fig. 1. The sample is assumed to be in perfect contact with the bottom and top plates and its lateral faces are free. The seismic mass is asymmetric in order to enforce shear motion in the sample and tuned so as to induce variations in the transfer functions in the frequency range of interest. In such manner, the estimation procedure is sensitive to both compression and shear moduli of the material. The specific properties of the seismic mass have to be chosen according to the foam to be tested and therefore preliminary tests are required in an experimental context. In fact, for a material with a reasonably low degree of anisotropy, the first mode visible in the transfer functions is associated with compression. Other types of motion such as shear or bending will have a stronger influence at higher frequencies. For the present purposes, measurements performed with this setup must clearly show these types of motion in the transfer functions.

The setup is located in a vacuum chamber in order to eliminate the influence of the fluid in the pores of the material. The data thus extracted consists of four transfer functions, between the displacement of the shaker, at point $\mathbf{r}_0$, and the displacement at points $\mathbf{r}_{ij}$ $(i, j = 1, 2)$. Using a cubic sample allows to repeat the experiment in the same conditions in the three directions of space, thus giving equal importance to all moduli in the estimation. The function $u(\omega_p, \mathbf{x}_0)$ of Eq. (18) is therefore numerically constructed as a juxtaposition of all 12 transfer functions. In such manner, each step of the optimization process takes into account the response of the sample in the three directions of space.

The simulations are performed using a conventional finite element package, integrated as a subroutine within the numerical optimization algorithm. The optimization procedure is implemented using the globally convergent method of moving asymptotes (GCMMA),[36] which is an improved version of the well-known method of moving asymptotes.[37]

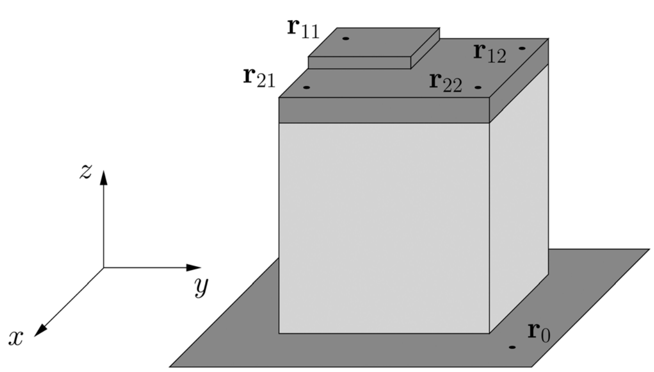

FIG. 1. Setup used for the inverse estimation. The foam sample, in light gray, is placed between a shaker and a seismic mass, in dark gray. The displacement is recovered at points $\mathbf{r}_0$ (reference) and $\mathbf{r}_{ij}$ $(i, j = 1, 2)$.

The latter is based on an approximation of the objective function in terms of variable-separated convex functions of the unknown parameters. At each iteration, the admissible bounds of each parameter, i.e., the asymptotes of the convex approximation, are updated and a unique optimal solution is retained. The improvement in GCMMA consists in performing outer iterations that ensure a strict decrease of the objective function and guarantees that the retained set of parameters is a feasible solution. A detailed presentation of the GCMMA algorithm would be out of scope here and the reader is referred to the original papers of Svanberg.[36,37]

IV. RESULTS

In this paragraph, the ability of the proposed method to estimate the physical parameters of a given material to a reasonable degree of accuracy is evaluated. In order to validate the method against a material with *a priori* known properties, the target to be reached is numerically generated using an arbitrary set of feasible physical parameters. The target material being fictitious, it should be noted that no further assumptions are made on the values of its parameters and thus the following discussion is merely a numerical example.

The simulations are performed on 40 linearly-spaced frequencies from 1 Hz to 500 Hz. The foam sample is cubic, with a side length of 10 cm and a density of 45 $\mathrm{kg\,m^{-3}}$. The parameters of the target material are chosen as

$$
\begin{aligned}
&E_1 = 120.9 \text{ kPa},\\
&E_2 = 84.5 \text{ kPa},\\
&E_3 = 135.2 \text{ kPa},\\
&G_{23} = 37 \text{ kPa},\\
&G_{31} = 60 \text{ kPa},\\
&G_{12} = 37.5 \text{ kPa},\\
&\nu_{12} = 0.279,\\
&\nu_{13} = 0.285,\\
&\nu_{23} = 0.282,\\
&\alpha = 0.855,\\
&\beta = 307.876 \text{ rad s}^{-1},\\
&b = 3.
\end{aligned}
\tag{19}
$$

The isotropic material of reference, used as a starting point for the optimizer, is arbitrary and as such the specific values of the parameters do not influence the optimal solution. This was verified by repeating the optimization process using randomly generated starting points. In the present example, the properties of the isotropic material of reference are arbitrarily chosen as

$$
\begin{aligned}
&E = 130 \text{ kPa},\\
&G = 50 \text{ kPa},\\
&\nu = 0.3,\\
&\alpha = 0.9,\\
&\beta = 314.159 \text{ rad s}^{-1},\\
&b = 2.5.
\end{aligned}
\tag{20}
$$

TABLE I. Values of the non-dimensional optimization variables for the targeted material and for the optimal solution.

$\mathbf{x}$	$\mathbf{x}_0$	$\mathbf{x}_{\mathrm{opt}}$	$\frac{\mathbf{x}_{\mathrm{opt}}-\mathbf{x}_0}{\mathbf{x}_0}$ (%)
$\xi 1$	0.93	0.933644	0.39
$\xi 2$	0.65	0.652820	0.43
$\xi 3$	1.04	1.044319	0.42
$\xi 4$	0.74	0.741094	0.15
$\xi 5$	1.2	1.202569	0.21
$\xi 6$	0.75	0.750598	0.08
$\xi 7$	0.93	0.901233	−3.09
$\xi 8$	0.95	0.972456	2.36
$\xi 9$	0.94	0.893167	−4.98
$\xi 10$	0.95	0.949772	−0.02
$\xi 11$	0.98	0.978668	−0.14
$\xi 12$	1.2	1.201572	0.13

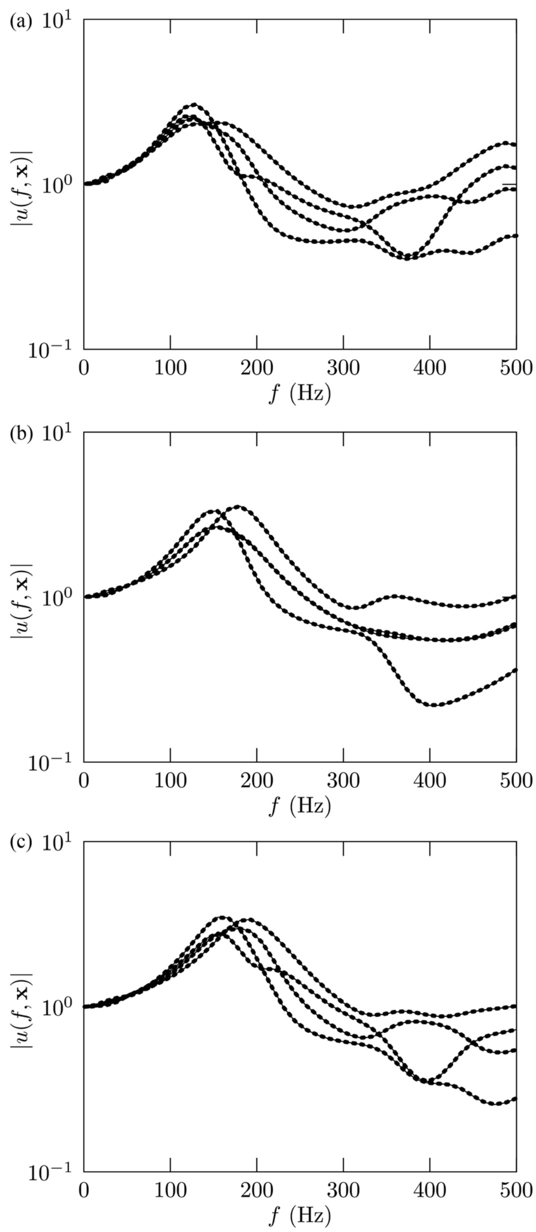

FIG. 2. Transfer functions between point $\mathbf{r}_0$ and the top plate points. (a) Foam sample oriented with x direction along the vertical, (b) y direction along the vertical, (c) z direction along the vertical. ——, target; - - - -, optimal solution.

The simulations are repeated orienting the material in the three directions of space. For each simulation, the top four transfer functions are extracted. The seismic mass is considered rigid and comprises two square plates, one on top of the other, as illustrated in Fig. 1. The first has dimensions 10 cm $\times$ 10 cm $\times$ 2 cm and density $100\,\mathrm{kg\,m^{-3}}$ and the second has dimensions 5 cm $\times$ 5 cm $\times$ 1 cm and density $800\,\mathrm{kg\,m^{-3}}$. The 12 transfer functions corresponding to measurements on the four top points in the three directions of space were used as the database for the GCMMA optimization algorithm. The latter is implemented in MATLAB and the finite element model is a COMSOL Multiphysics subroutine providing the transfer functions. Using quadratic Lagrange tetrahedral elements, the numerical model convergence is reached with a total of 2604 degrees of freedom. The optimizer is considered to have converged to an optimal set of parameters when the variation of the objective function and of each one of the parameters is less than 10^{-3}.

The values of the optimization space for the target, initial and optimal sets are summarized in Table I, together with the relative error between the optimal and target sets of parameters.

Figure 2 shows the transfer functions of the setup for the three directions of space. The solid line represents the transfer functions of the targeted material, i.e., the responses of the setup with properties $\mathbf{x}_0$. The dashed line represents the optimal transfer functions obtained with the present method. It can be observed that the optimized set of parameters leads to a good agreement of the transfer functions. Figure 3 shows the objective function and the values of the optimization variables at each iteration of the optimizer. In the example here presented, the optimization space reaches a stable state after 32 iterations of the GCMMA algorithm.

Figure 4 shows the stiffness matrix, computed with the optimal and target values of the optimization variables. The mean absolute relative error between the optimal and target stiffness matrices is given by

$$\frac{1}{N}\sum_{p=1}^{N}\left|\frac{H_{ij}(\omega_p,\mathbf{x})-H_{ij}(\omega_p,\mathbf{x}_0)}{H_{ij}(\omega_p,\mathbf{x}_0)}\right| = \begin{bmatrix} 0.53 & 4.04 & 0.90 & \emptyset & \emptyset & \emptyset \\ 4.04 & 1.36 & 4.75 & \emptyset & \emptyset & \emptyset \\ 0.90 & 4.75 & 0.25 & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & 0.15 & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & 0.22 & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & 0.09 \end{bmatrix} (\%). \tag{21}$$

From Table I and Eq. (21) it can be observed that the present method provides a reasonable agreement between the obtained and targeted material properties.

V. DISCUSSION

The inverse estimation presented above is performed on a fictitious material with an arbitrary set of properties, within the feasible ranges set by Eq. (11). For the example chosen,

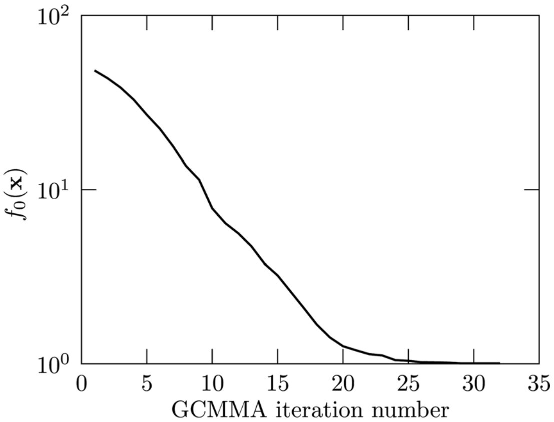

FIG. 3. Convergence of the objective function with the iterations of the optimizer.

the parameters of the model are obtained within a reasonable degree of accuracy. This inverse estimation procedure is intended to have a broad application range. However, some aspects should be considered for its application to a particular case.

It should be noted that the good agreement between the optimal solution and the targeted material is reached by the optimizer simultaneously in the three directions of space. In fact, the robustness of the objective function, Eq. (18), lies on the fact that it includes the 12 transfer functions (four measurement points in the three directions of space) at each step of the optimization. Therefore, equal importance is given to the moduli associated to the different directions of space. In this manner, the accuracy in the estimation of the different moduli is linked to their nature and not to their intrinsic predominance in one particular direction. Such predominance explains the insufficiency of uniaxial material tests[11] for anisotropic cases.

Furthermore, the sensitivity of the method to the different types of moduli (E_i, G_{ij} or ν_{ij}) is governed by the loading. The asymmetry of the seismic mass controls the type of motion that is predominant in the response. As pointed out in Sec. III B, for the method to be applicable to a given material, the transfer functions recovered from the sample must exhibit the types of motion associated with the different moduli. Therefore, the material to be tested being *a priori* unknown, preliminary tests are necessary for each particular material in order to choose an appropriate loading.

The method presented herein is in principle not restricted to a particular frequency range. However, at sufficiently high

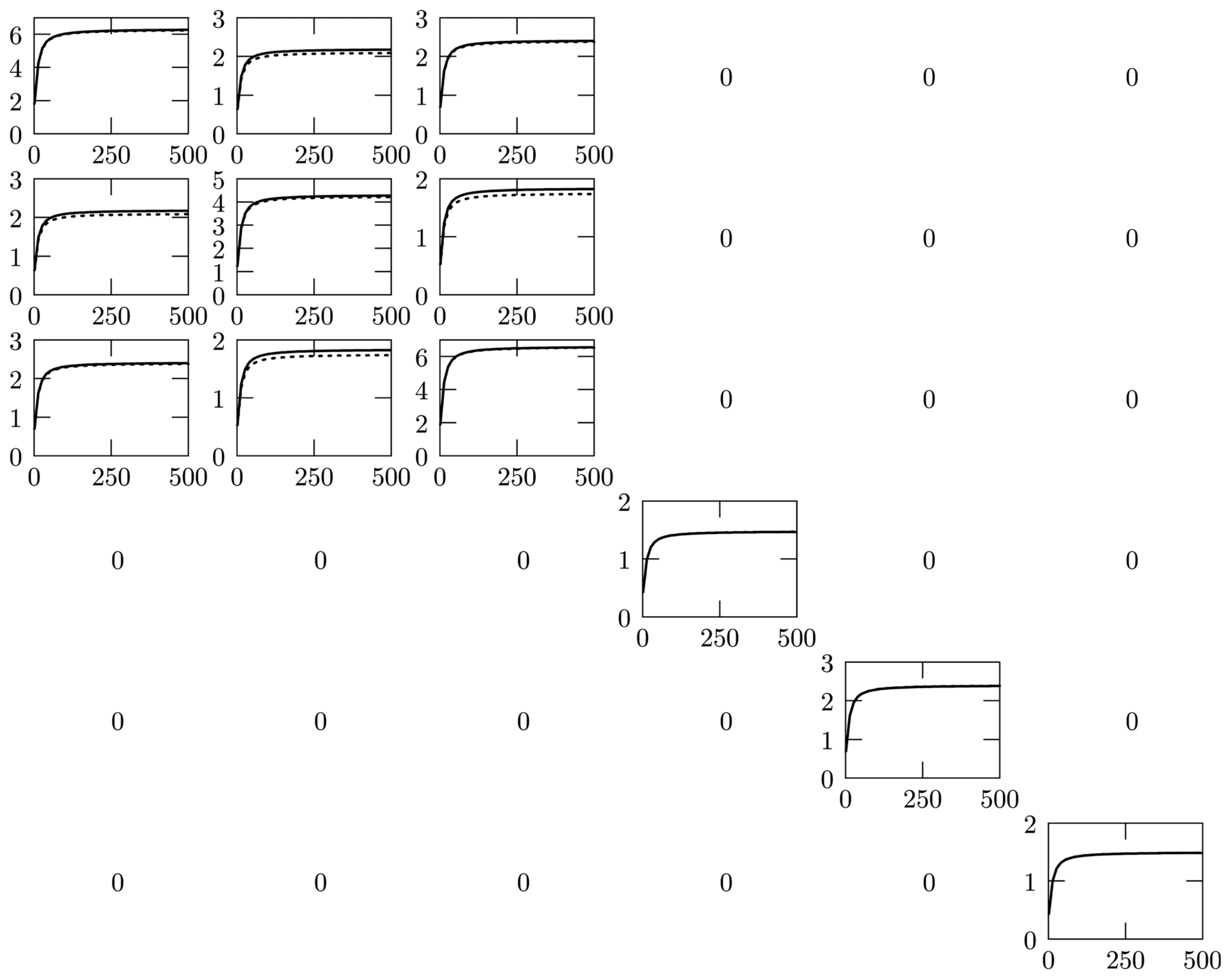

FIG. 4. Terms of the stiffness matrix, in 10^5 Pa, as a function of frequency, in Hz. Solid line, stiffness matrix of the target material; dashed line, stiffness matrix computed with the optimal set of parameters.

frequencies or in the case of materials with high damping, the response of a particular material may exhibit a less resonant behavior. In such cases, the variations of the transfer functions with frequency may conceal the contribution of anisotropy.

VI. CONCLUSION

In this paper a methodology for the inverse estimation of the elastic and anelastic parameters governing the dynamic behavior of the porous frame of open-cell foams is presented. The underlying model of the material is based on the augmented Hooke's law, which provides a clear distinction between the elastic and anelastic properties of the frame. The fractional-derivative formalism used for describing anelasticity provides a description of the energy dissipation in the material with a low number of parameters to estimate. In addition, introducing an equivalent isotropic medium as a starting point for the parameter estimation allows for the use of the present approach as a means of refining results obtained by any existing method for isotropic media. Thus, the estimated parameters can be interpreted as the degrees of anisotropy in the different moduli.

The proposed model is general enough to account for different types of anisotropy in the elastic and anelastic properties. However, for the purposes of ascertaining the feasibility of the proposed approach, the number of parameters to estimate was reduced by restricting the analysis to materials presenting proportional damping. The inverse estimation procedure, consisting in minimizing the deviation between the simulated and the target transfer functions, is implemented as an optimization routine, based on the globally convergent method of moving asymptotes. The latter has the advantage, over other optimization algorithms, to provide a fast convergence of a problem involving a large number of unknowns.

The application of the proposed method to a cubic sample of material shows that a good agreement is reached between the optimal set of parameters and the targeted ones. The setup that has been used is particularly sensitive to compression and shear motion, such that the estimated Young's moduli and shear moduli are within 0.5% of error, while the estimated Poisson ratios are found within an error margin of 5%.

In an experimental situation, the choice of the different loading cases used to balance the sensitivity of the measurement to different types of motion still remains an open question. Also, special attention should be given to the boundary conditions as these may differ between the experiment and the numerical model. In particular, the attachment at the interface between the sample and the seismic mass may introduce additional mass and damping. As stated in the description of the method, the setup used in the present paper is intended to be easily implementable in practice. For example, another possible setup could consist of a single material orientation with both axial and shear excitations.

The experimental characterization of open-cell foams using the presented method is currently under development. The data thus obtained can be directly employed for simulating the vibroacoustic behavior of anisotropic porous foams using the formulation previously developed by Hörlin and Göransson.[31]

[1]A. T. Huber and L. J. Gibson, "Anisotropy of foams," J. Mater. Sci. **23**, 3031–3040 (1988).
[2]W. E. Warren and A. M. Kraynik, "The linear elastic properties of open-cell foams," J. Appl. Mech. **55**, 341–346 (1988).
[3]A. Geslain, O. Dazel, J.-P. Groby, S. Sahraoui, and W. Lauriks, "Influence of static compression on mechanical parameters of acoustic foams," J. Acoust. Soc. Am. **130**, 818–825 (2011).
[4]H. J. Rice and P. Göransson, "A dynamical model of light fibrous materials," Int. J. Mech. Sci. **41**, 561–579 (1999).
[5]P. Göransson and P. Lemarinier, "Sound transmission through double-panel systems lined with soaked insulation materials," in *Fourth AIAA/CEAS Aeroacoustics Conference*, Paper No. AIAA–98-2345 (Toulouse, France) (2–4 June 1998).
[6]M. Melon, E. Mariez, C. Ayrault, and S. Sahraoui, "Acoustical and mechanical characterization of anisotropic open-cell foams," J. Acoust. Soc. Am. **104**, 2622 (1998).
[7]E. Mariez, S. Sahraoui, and J. F. Allard, "Elastic constants of polyurethane foam's skeleton for biot model," in *Internoise 96*, 951–954 (Institute of Acoustics, Liverpool, UK, 1996).
[8]S. Sahraoui, E. Mariez, and M. Etchessahar, "Linear elastic properties of anisotropic open-cell foams," J. Acoust. Soc. Am. **110**, 635 (2001).
[9]S. Sahraoui, E. Mariez, and M. Etchessahar, "Mechanical testing of polymeric foams at low frequency," Polym. Test. **20**, 93–96 (2000).
[10]L. Jaouen, A. Renault, and M. Deverge, "Elastic and damping characterizations of acoustical porous materials: Available experimental methods and applications to a melamine foam," Appl. Acoust. **69**, 1129–1140 (2008).
[11]T. Pritz, "Transfer function method for investigating the complex modulus of acoustic materials: spring-like specimen," J. Sound Vib. **72**, 317–341 (1980).
[12]T. Pritz, "Transfer function method for investigating the complex modulus of acoustic materials: Rod-like specimen," J. Sound Vib. **81**, 359–376 (1982).
[13]T. Pritz, "Frequency dependence of frame dynamic characteristics of mineral and glass wool materials," J. Sound Vib. **106**, 161–169 (1986).
[14]T. Pritz, "Dynamic Young's modulus and loss factor of plastic foams for impact sound isolation," J. Sound Vib. **178**, 315–322 (1994).
[15]M. A. Biot, "Theory of stress-strain relations in anisotropic viscoelasticity and relaxation phenomena," J. Appl. Phys. **25**, 1385–1391 (1954).
[16]K. Dovstam and M. Dalenbring, "Damping function estimation based on modal receptance models and neural nets," Comput. Mech. **19**, 271–286 (1997).
[17]M. Dalenbring, "Damping function estimation based on measured vibration frequency responses and finite-element displacement modes," Mech. Syst. Sig. Process. **13**, 547–569 (1999).
[18]M. Dalenbring, "An explicit formulation of a three-dimensional material damping model with transverse isotropy," Int. J. Solids Struct. **39**, 225–249 (2002).
[19]M. Dalenbring, "Experimental material damping estimation for planar isotropic laminate structures," Int. J. Solids Struct. **39**, 5053–5079 (2002).
[20]R. L. Bagley and P. J. Torvik, "A theoretical basis for the application of fractional calculus to viscoelasticity," J. Rheol. **27**, 201 (1983).
[21]R. L. Bagley and P. J. Torvik, "Fractional calculus—A different approach to the analysis of viscoelastically damped structures" AIAA J. **21**, 741–748 (1983).
[22]K. Dovstam, "Augmented Hooke's law based on alternative stress relaxation models," Comput. Mech. **26**, 90–103 (2000).
[23]M. Caputo and F. Mainardi, "A new dissipation model based on memory mechanism," Pure Appl. Geophys. **91**, 134–147 (1971).
[24]M. Caputo and F. Mainardi, "Linear models of dissipation in anelastic solids," Riv. Nuovo Cimento **1**, 161–198 (1971).
[25]T. Pritz, "Analysis of four-parameter fractional derivative model of real solid materials," J. Sound Vib. **195**, 103–115 (1996).
[26]T. Pritz, "Five-parameter fractional derivative model for polymeric damping materials," J. Sound Vib. **265**, 935–952 (2003).
[27]S.-Y. Kim and D.-H. Lee, "Identification of fractional-derivative-model parameters of viscoelastic materials from measured FRFs," J. Sound Vib. **324**, 570–586 (2009).
[28]A. L. Araújo, C. M. M. Soares, C. A. M. Soares, and J. Herskovits, "Optimal design and parameter estimation of frequency dependent viscoelastic laminated sandwich composite plates" Comp. Struct. **92**, 2321–2327 (2010).
[29]M. A. Biot, "Theory of deformation of a porous viscoelastic anisotropic solid," J. Appl. Phys. **27**, 459–467 (1956).

[30] M. A. Biot, "Mechanics of deformation and acoustic propagation in porous media," J. Appl. Phys. **33**, 1482–1498 (1962).
[31] N. E. Hörlin and P. Göransson, "Weak, anisotropic symmetric formulations of Biot's equations for vibro-acoustic modelling of porous elastic materials," Int. J. Numer. Methods Eng. **84**, 1519–1540 (2010).
[32] R. M. Jones, *Mechanics of Composite Materials* (Taylor and Francis, Philadelphia, PA, 1999), pp. 519.
[33] O. Doutres, N. Dauchez, J.-M. Génevaux, and G. Lemarquand, "On the use of a loudspeaker for measuring the viscoelastic properties of sound absorbing materials," J. Acoust. Soc. Am. **124**, EL335–EL340 (2008).
[34] A. N. Norris, "The isotropic material closest to a given anisotropic material," J. Mech. Mater. Struct. **1**, 223–238 (2006).
[35] J. Rodríguez Ahlquist, "Porous foam characterization and simulation," Master's thesis, Royal Institute of Technology, Stockholm, Sweden, 2002, pp. 92.
[36] K. Svanberg, "A class of globally convergent optimization methods based on conservative convex separable approximations," SIAM J. Optim. **12**, 555 (2002).
[37] K. Svanberg, "The method of moving asymptotes—A new method for structural optimization," Int. J. Numer. Methods Eng. **24**, 359–373 (1987).